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The transformation properties of vector multipole fields under a translation of coordinate origin

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Abstract. Addition theorems for Stratton's vector multipole fields are derived, exhibiting the results of Stein and Cruzan in a more compact and useful form.

1. Introduction

The vector multipole fields (Stratton 1941)

$$\begin{aligned} \boldsymbol{L}_{lm}(\boldsymbol{r}) &= \boldsymbol{\nabla}\phi_{lm}(\boldsymbol{r}) \\ \boldsymbol{M}_{lm}(\boldsymbol{r}) &= \boldsymbol{\nabla}\times\boldsymbol{r}\phi_{lm}(\boldsymbol{r}) \\ \boldsymbol{N}_{lm}(\boldsymbol{r}) &= k^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}_{lm}(\boldsymbol{r}) \\ \phi_{lm}(\boldsymbol{r}) &= j_l(kr)C_{lm}(\theta,\varphi), \end{aligned} \tag{1}$$

where $j_l(kr)$ is a spherical Bessel function and $C_{lm}(\theta, \varphi)$ is a spherical harmonic normalized to $C_{l0}(0, 0) = 1$ (Brink and Satchler 1968), have simple and well known transformation properties under rotation. Their transformation properties under translation of the coordinate frame to which they are referred have been given by Stein (1961) and Cruzan (1962) but their expressions are rather cumbersome. As these transformation properties are important in the discussion of many problems involving the electromagnetic field it seems useful to present them in a simple and more compact form.

Stein (1961) and Cruzan (1962) adopted the procedure of writing

$$\boldsymbol{\nabla}\times\boldsymbol{r}\phi_{lm}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{R}\phi_{lm}(\boldsymbol{r}) + \boldsymbol{\nabla}\times\boldsymbol{r}'\phi_{lm}(\boldsymbol{r}),$$

where $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}'$, expanding the scalar field $\phi_{lm}(\boldsymbol{r})$ in terms of $\phi_{st}(\boldsymbol{R})$ and $\phi_{l'm'}(\boldsymbol{r}')$ (Friedmann and Russek 1954) and simplifying the resulting expressions by the use of various recurrence relations obeyed by the spherical Bessel and harmonic functions. As might be expected such a method cannot fully exploit the group theoretical properties (Talman 1968) of the functions involved and their expressions are consequently rather cumbersome. Here we use a method which is an extension of that of Danos and Maximon (1965) to derive the transformation coefficients in a relatively concise and transparent form. We begin by reviewing the basic ideas and notation.

The vector Helmholtz equation (Morse and Feshbach 1953)

$$(\nabla^2 + k^2)\boldsymbol{X}(\boldsymbol{r}) = 0 \tag{2}$$

where $\boldsymbol{X}(\boldsymbol{r})$ is a vector field and k^2 is a positive real constant, occurs in the discussion of many physical problems. A solution of such a vector differential equation will be specified by a set of these quantities corresponding to the three components of the vector field in some coordinate system. It is only when the vector field is referred to a Cartesian coordinate system that each of the three vector components satisfies a scalar Helmholtz equation. When the differential equation is expressed in terms of any other coordinate system the resulting equations involving the components of the vector field in that coordinate system are coupled (Morse and Feshbach 1953) and are consequently much more difficult to solve. It is possible, however, by exploiting the invariance properties of the Helmholtz operator, to deduce the form of a set of vector solutions directly from a solution of the scalar equation.

It is readily verified that the Helmholtz operator ∇^2+k^2 is invariant under both the translation and rotation of the coordinate frame with respect to which it is defined. Thus we may write

$$\nabla'^2 \equiv \nabla^2, \qquad \boldsymbol{r}' = \boldsymbol{a} + \boldsymbol{r}$$

and

$$\nabla''^2 \equiv \nabla^2, \qquad \boldsymbol{r}'' = \mathbf{T}\boldsymbol{r}.$$

Here $\boldsymbol{a}$ is a vector defining the translation of the coordinate frame while $\mathbf{T}$ is an orthogonal matrix defining its rotation. If $f(\boldsymbol{r})$ is a scalar solution of the Helmholtz equation then

$$(\nabla^2+k^2)f(\boldsymbol{r}) = (\nabla'^2+k^2)f(\boldsymbol{r}) = (\nabla^2+k^2)f(\boldsymbol{r}-\boldsymbol{a}) = 0$$

$$(\nabla^2+k^2)f(\boldsymbol{r}) = (\nabla''^2+k^2)f(\boldsymbol{r}) = (\nabla^2+k^2)f(\tilde{\mathbf{T}}\boldsymbol{r})$$

where $\tilde{\mathbf{T}}$ is the transpose of $\mathbf{T}$. Expanding $f(\boldsymbol{r}-\boldsymbol{a})$ and $f(\tilde{\mathbf{T}}\boldsymbol{r})$ to first order in Taylor series we have

$$(\nabla^2+k^2)(f(\boldsymbol{r}) - \boldsymbol{a} \cdot \boldsymbol{\nabla} f(\boldsymbol{r})) = 0$$

$$(\nabla^2+k^2)(f(\boldsymbol{r}) + \theta \boldsymbol{n} \cdot \boldsymbol{\nabla} \times \boldsymbol{r} f(\boldsymbol{r})) = 0;$$

$\boldsymbol{n}$ is the unit vector along the axis of rotation; θ is the angle of rotation about that axis. From these equations it follows that $\boldsymbol{\nabla} f(\boldsymbol{r})$ and $\boldsymbol{\nabla}\times\boldsymbol{r} f(\boldsymbol{r})$ are solutions of (2). It is further readily verified that if $\boldsymbol{X}(\boldsymbol{r})$ is a divergenceless solution of (2) then $\boldsymbol{\nabla}\times\boldsymbol{X}(\boldsymbol{r})$ is also a solution of that equation. In this way we have obtained a set of three vector solutions from one scalar solution. So far no mention has been made of the coordinate system to which these solutions are referred. The scalar Helmholtz equation, separated in the spherical polar coordinate system, has solutions of the form

$$\psi_{lm}(\boldsymbol{r}) = z_l(kr)C_{lm}(\theta, \varphi)$$

where now $z_l(kr)$ is a spherical Bessel or Neumann function. These solutions have the unique merit of forming irreducible sets under the operator $\boldsymbol{\nabla}\times\boldsymbol{r}$ and, furthermore, of having very special properties under the operator $\boldsymbol{\nabla}$. This is quite readily understood in the context of the theory of representations of the three-dimensional Euclidean group (Talman 1968). The functions $\psi_{lm}(\boldsymbol{r})$ play the role of partner functions to those representations of this group which are themselves irreducible representations of its rotation subgroup. As is discussed by Talman such partner functions obey sets of coupled differential equations whose forms are determined by the parametrization used to specify the group elements, and by the so called product function which maps the group multiplication property onto the parameter domain. A lengthy and rather involved calculation, paralleling that of Talman's treatment of the two-dimensional Euclidean group, shows that for the translation subgroup, of which the components of the gradient operator are the infinitesimal generators, these equations take the form

$$\nabla z_l(kr)C_{lm}(\theta,\varphi) = k\{[(l+1)/(2l+1)]^{1/2} z_{l+1}(kr)\mathbf{C}^m_{ll+1}(\theta,\varphi) + [l/(2l+1)]^{1/2} z_{l-1}(kr)\mathbf{C}^m_{ll-1}(\theta,\varphi)\} \tag{3}$$

where

$$\mathbf{C}^m_{l\lambda}(\theta,\varphi) = (-1)^{\lambda-1+m}(2\lambda+1)^{1/2}\sum_\mu \boldsymbol{\xi}_\mu C_{\lambda,m-\mu}(\theta,\varphi)\begin{pmatrix} \lambda & 1 & l \\ m-\mu & \mu & -m \end{pmatrix}. \tag{4}$$

Here $\boldsymbol{\xi}_\mu$ are the spherical basis vectors given, in the Condon and Shortley phase convention, by

$$\begin{pmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_0 \\ \boldsymbol{\xi}_{-1} \end{pmatrix} = 2^{-1/2}\begin{pmatrix} -1 & -\mathrm{i} & 0 \\ 0 & 0 & 2^{1/2} \\ 1 & -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{e}_X \\ \boldsymbol{e}_Y \\ \boldsymbol{e}_Z \end{pmatrix} \tag{5}$$

and $\left(\begin{smallmatrix} a & b & c \\ \alpha & \beta & \gamma \end{smallmatrix}\right)$ is a Wigner $3j$-symbol.

If now the following recurrence relations for the spherical Bessel functions:

$$\left(\frac{l}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)z_l(r) = z_{l+1}(r)$$

$$\left(\frac{l+1}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)z_l(r) = z_{l-1}(r)$$

are substituted in the result (3) one can deduce the so called gradient formula (Rose 1957)

$$\nabla f(\boldsymbol{r})C_{lm}(\theta,\varphi) = \left(\frac{l+1}{2l+1}\right)^{1/2}\left(\frac{l}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\mathbf{C}^m_{ll+1}(\theta,\varphi) + \left(\frac{l}{2l+1}\right)^{1/2}\left(\frac{l+1}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\mathbf{C}^m_{ll-1}(\theta,\varphi). \tag{6}$$

This discussion provides some insight into the relatively simple form of (6), which is obtained in conventional derivations (Rose 1957) only after a fairly complicated piece of Racah algebra. The corresponding equations for the rotation subgroup of which the components of $\nabla\times\boldsymbol{r}$ are the infinitesimal generators, are the shift operator equations familiar from angular momentum theory. In the notation used here these take the form

$$\nabla\times\boldsymbol{r}z_l(kr)C_{lm}(\theta,\varphi) = -\mathrm{i}[l(l+1)]^{1/2}z_l(kr)\mathbf{C}^m_{ll}(\theta,\varphi). \tag{7}$$

It is shown in the appendix that

$$k^{-1}\nabla\times\nabla\times\boldsymbol{r}z_l(kr)C_{lm}(\theta,\varphi) = [l(l+1)]^{1/2}\{[(l+1)/(2l+1)]^{1/2}z_{l-1}(kr)\mathbf{C}^m_{ll-1}(\theta,\varphi) - [l/(2l+1)]^{1/2}z_{l+1}(kr)\mathbf{C}^m_{ll+1}(\theta,\varphi)\}. \tag{8}$$

Consequently the vector fields (1) have simple analytic forms and provide a very convenient basis for the expansion of vector solutions of the Helmholtz equation and its inhomogenous analogues.

2. The translation properties of the vector multipole fields

As the vector fields (1) form a complete set for the expansion of solutions of the vector Helmholtz equation (Stratton 1941), which is itself translationally invariant, we can now express a multipole field referred to one origin in terms of those referred to a translated origin. Furthermore the fields $\boldsymbol{L}_{lm}(\boldsymbol{r})$ are sufficient to provide an expansion of an irrotational field while $\boldsymbol{M}_{lm}(\boldsymbol{r})$, $\boldsymbol{N}_{lm}(\boldsymbol{r})$ can provide an expansion of any divergenceless solution. Thus we may write

$$\begin{aligned}\boldsymbol{L}_{lm}(\boldsymbol{r}) &= \sum_{l',m'} A^{(L,L)}_{lm,l'm'}(\boldsymbol{R})\boldsymbol{L}_{l'm'}(\boldsymbol{r}')\\ \boldsymbol{M}_{lm}(\boldsymbol{r}) &= \sum_{l',m'} (A^{(M,M)}_{lm,l'm'}(\boldsymbol{R})\boldsymbol{M}_{l'm'}(\boldsymbol{r}') + A^{(M,N)}_{lm,l'm'}(\boldsymbol{R})\boldsymbol{N}_{l'm'}(\boldsymbol{r}'))\\ \boldsymbol{N}_{lm}(\boldsymbol{r}) &= \sum_{l',m'} (A^{(N,M)}_{lm,l'm'}(\boldsymbol{R})\boldsymbol{M}_{l'm'}(\boldsymbol{r}') + A^{(N,N)}_{lm,l'm'}(\boldsymbol{R})\boldsymbol{N}_{l'm'}(\boldsymbol{r}'))\end{aligned} \tag{9}$$

where $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{R}$.

As $\boldsymbol{N}_{lm}(\boldsymbol{r}) = k^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}_{lm}(\boldsymbol{r})$ is a translationally invariant relation the following identities hold:

$$\begin{aligned}A^{(M,M)}_{lm,l'm'}(\boldsymbol{R}) &= A^{(N,N)}_{lm,l'm'}(\boldsymbol{R}) \equiv A_{lm,l'm'}(\boldsymbol{R})\\ A^{(M,N)}_{lm,l'm'}(\boldsymbol{R}) &= A^{(N,M)}_{lm,l'm'}(\boldsymbol{R}) \equiv A'_{lm,l'm'}(\boldsymbol{R}).\end{aligned} \tag{10}$$

Of these coefficients the $A^{(L,L)}_{lm,l'm'}(\boldsymbol{R})$ are obtained most readily. As the gradient operator is translationally invariant they can be obtained immediately from the translational properties of the scalar multipole field $j_l(kr)C_{lm}(\theta, \varphi)$ (Friedmann and Russek 1954, Talman 1968):

$$\begin{aligned}j_l(kr)C_{lm}(\theta,\varphi) = \sum_{l',s,m'} &\mathrm{i}^{l+l'-s}(-1)^m(2l'+1)(2s+1)\begin{pmatrix} l & s & l' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & s & l' \\ m & m'-m & -m'\end{pmatrix}\\ &\times j_{l'}(kr')C_{l'm'}(\theta',\varphi')j_s(kR)C_{sm-m'}(\theta_R,\varphi_R).\end{aligned} \tag{11}$$

This yields

$$\begin{aligned}&\boldsymbol{\nabla} j_l(kr)C_{lm}(\theta,\varphi)\\ &\quad= \sum_{l',s,m'} \mathrm{i}^{l+l'-s}(-1)^m(2l'+1)(2s+1)\begin{pmatrix} l & s & l' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & s & l' \\ m & m'-m & -m'\end{pmatrix}\\ &\qquad\times j_s(kR)C_{sm-m'}(\theta_R,\varphi_R)\boldsymbol{\nabla}' j_{l'}(kr')C_{l'm'}(\theta',\varphi')\end{aligned} \tag{12}$$

and hence

$$\begin{aligned}A^{(L,L)}_{lm,l'm'}(\boldsymbol{R}) = \sum_s &\mathrm{i}^{l+l'-s}(-1)^m(2l'+1)(2s+1)\begin{pmatrix} l & s & l' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & s & l' \\ m & m'-m & -m'\end{pmatrix}\\ &\times j_s(kR)C_{sm-m'}(\theta_R,\varphi_R).\end{aligned} \tag{13}$$

The presence of the $3j$-symbols in this expression limits the effective range of s to $|l-l'|\leqslant s\leqslant l+l'$. From (7) we have

$$\boldsymbol{M}_{lm}(\boldsymbol{r})=\mathrm{i}[l(l+1)(2l+1)]^{1/2}(-1)^{l+m}\sum_{\mu}\xi_{\mu}C_{lm-\mu}(\theta,\varphi)\begin{pmatrix} l & 1 & l\\ m-\mu & \mu & -m\end{pmatrix}j_l(kr).$$

Using (11) this becomes

$$\boldsymbol{M}_{lm}(\boldsymbol{r})=[l(l+1)(2l+1)]^{1/2}(-1)^{l+m}\sum_{\substack{l',s'\\ \mu,t}}(-1)^{m-\mu}\,\mathrm{i}^{l+l'-s+1}(2l'+1)(2s+1)$$
$$\times\begin{pmatrix} l & s & l'\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & 1 & l\\ m-\mu & \mu & -m\end{pmatrix}\begin{pmatrix} l & s & l'\\ m-\mu & -t & t+\mu-m\end{pmatrix}\xi_{\mu}$$
$$\times j_l(kr')C_{l'm-t-\mu}(\theta',\varphi')j_s(kR)C_{st}(\theta_R,\varphi_R). \qquad (14)$$

The product of the two $3j$-symbols can be recoupled to give

$$\begin{pmatrix} l & 1 & l\\ m-\mu & \mu & -m\end{pmatrix}\begin{pmatrix} l & s & l'\\ m-\mu & -t & t+\mu-m\end{pmatrix}$$
$$=\sum_{f}(-1)^{1+l'+s+f+m-\mu-t}(2f+1)\begin{Bmatrix} 1 & l & l\\ s & l' & f\end{Bmatrix}$$
$$\times\begin{pmatrix} l' & 1 & f\\ m-t-\mu & \mu & t-m\end{pmatrix}\begin{pmatrix} l & s & f\\ -m & t & m-t\end{pmatrix}$$

where $\{\begin{smallmatrix} 1 & l & l\\ s & l' & f\end{smallmatrix}\}$ is a Wigner $6j$-symbol, which on substitution into (14) gives

$$\boldsymbol{M}_{lm}(\boldsymbol{r})=[l(l+1)(2l+1)]^{1/2}\sum_{\substack{l',s',f\\ \mu,m'}}(-1)^{1+f+m'}\,\mathrm{i}^{l+l'-s+1}(2l'+1)(2s+1)\begin{pmatrix} l & s & l'\\ 0 & 0 & 0\end{pmatrix}$$
$$\times\begin{Bmatrix} 1 & l & l\\ s & l' & f\end{Bmatrix}\begin{pmatrix} l' & 1 & f\\ m'-\mu' & \mu & -m'\end{pmatrix}\begin{pmatrix} l & s & f\\ -m & m-m' & m'\end{pmatrix}$$
$$\times\xi_{\mu}C_{l'm'-\mu}(\theta',\varphi')j_{l'}(kr')C_{sm-m'}(\theta_R,\varphi_R)j_s(kR). \qquad (15)$$

On identifying $\mathbf{C}_{fl'}^{m'}(\theta',\varphi')$ through its definition (4) this reduces to

$$\boldsymbol{M}_{lm}(\boldsymbol{r})=\sum_{fl'm'}B_{lm,fl'm'}(\boldsymbol{R})j_{l'}(kr')\mathbf{C}_{fl'}^{m'}(\theta',\varphi') \qquad (16)$$

the coefficient $B_{lm,fl'm'}(\boldsymbol{R})$ being given by

$$B_{lm,fl'm'}(\boldsymbol{R})=(-1)^f\,\mathrm{i}^{l-l'-s+1}[l(l+1)(2l+1)(2l'+1)]^{1/2}$$
$$\times\sum_{s}(2s+1)\begin{pmatrix} l & s & l'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} 1 & l & l\\ s & l' & f\end{Bmatrix}\begin{pmatrix} l & s & f\\ -m & m-m' & m'\end{pmatrix}$$
$$\times\, j_s(kR)C_{s,m-m'}(\theta_R,\varphi_R).$$

From the condition that $\boldsymbol{M}_{lm}(\boldsymbol{r})$ is divergenceless we obtain, on using (A.2) and the recurrence relations for the spherical Bessel functions:

$$B_{lm,ff+1\,m'}=-[f/(f+1)]^{1/2}B_{lm,ff-1\,m'}.$$

This result, in conjunction with (8), (9) and (16) allows us to write

$$\boldsymbol{M}_{lm}(\boldsymbol{r})=\sum_{l',m'}A_{lm,l'm'}(\boldsymbol{R})\boldsymbol{M}_{l'm'}(\boldsymbol{r}')+A'_{lm,l'm'}(\boldsymbol{R})\boldsymbol{N}_{l'm'}(\boldsymbol{r}')$$

$$\boldsymbol{N}_{lm}(\boldsymbol{r})=\sum_{l',m'}A'_{lm,l'm'}(\boldsymbol{R})\boldsymbol{M}_{l'm'}(\boldsymbol{r}')+A_{lm,l'm'}(\boldsymbol{R})\boldsymbol{N}_{l'm'}(\boldsymbol{r}')$$

where

$$A_{lm,l'm'}(\boldsymbol{R})=-\mathrm{i}^{l+l'}[l(l+1)(2l+1)(2l'+1)/l'(l'+1)]^{1/2}\times\sum_{s=|l-l'|}^{l+l'}\mathrm{i}^{-s}(2s+1)\begin{pmatrix}l&s&l'\\0&0&0\end{pmatrix}\begin{Bmatrix}1&l&l\\s&l'&l'\end{Bmatrix}\begin{pmatrix}l&s&l'\\-m&m-m'&m'\end{pmatrix}\times j_s(kR)C_{sm-m'}(\theta_R,\varphi_R)\tag{17}$$

and

$$A'_{lm,l'm'}(\boldsymbol{R})=-\mathrm{i}^{l+l'}(l'+1)^{-1}[l(l+1)(2l+1)(2l'+1)(2l'-1)/l']^{1/2}\times\sum_{s=|l-l'|}^{l+l'}\mathrm{i}^{-s}(2s+1)\begin{pmatrix}l&s&l'-1\\0&0&0\end{pmatrix}\begin{Bmatrix}1&l&l\\s&l'-1&l'\end{Bmatrix}\begin{pmatrix}l&s&l'\\-m&m-m'&m'\end{pmatrix}\times C_{sm-m'}(\theta_R,\varphi_R)j_s(kR).\tag{18}$$

s runs over values for which the $3j$-symbols do not vanish.

It is also possible to expand the vector multipole fields derived from spherical Hankel, rather than Bessel, functions in one frame in terms of the fields $\boldsymbol{M}_{lm}(\boldsymbol{r})$, $\boldsymbol{N}_{lm}(\boldsymbol{r})$, $\boldsymbol{L}_{lm}(\boldsymbol{r})$ in another, translated, frame at whose origin these fields will be regular. Whereas the expressions derived above converge for all values of $|\boldsymbol{r}'|$ and $|\boldsymbol{R}|$ the corresponding series in the Hankel function case converge only when $|\boldsymbol{r}'|<|\boldsymbol{R}|$.

Defining the vector fields

$$\boldsymbol{L}'_{lm}(\boldsymbol{r})=\boldsymbol{\nabla}h_l(kr)C_{lm}(\theta,\varphi),\qquad \boldsymbol{M}'_{lm}(\boldsymbol{r})=\boldsymbol{\nabla}\times\boldsymbol{r}h_l(kr)C_{lm}(\theta,\varphi)\tag{1a}$$

$\boldsymbol{N}_{lm}(\boldsymbol{r})=k^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}_{lm}(\boldsymbol{r})$, where $h_l(kr)$ is a spherical Hankel function of order l, and using the addition theorem

$$h_l(kr)C_{lm}(\theta,\varphi)=\sum_{l',s,t}\mathrm{i}^{l'+l-s}(-1)^m(2l'+1)(2s+1)\begin{pmatrix}l&s&l'\\0&0&0\end{pmatrix}\begin{pmatrix}l&s&l'\\m&-t&t-m\end{pmatrix}\times j_{l'}(kr')C_{l'm-t}(\theta',\varphi')h_s(kR)C_{st}(\theta_R,\varphi_R)\tag{11a}$$

for $|\boldsymbol{r}'|<|\boldsymbol{R}|$, $\boldsymbol{r}=\boldsymbol{r}'+\boldsymbol{R}$, we have

$$\boldsymbol{M}'_{lm}(\boldsymbol{r})=\sum_{l',m'}a_{lm,l'm'}(\boldsymbol{R})\boldsymbol{M}_{l'm'}(\boldsymbol{r}')+a'_{lm,l'm'}(\boldsymbol{R})\boldsymbol{N}_{l'm'}(\boldsymbol{r}')$$

$$\boldsymbol{N}'_{lm}(\boldsymbol{r})=\sum_{l',m'}a'_{lm,l'm'}(\boldsymbol{R})\boldsymbol{M}_{l'm'}(\boldsymbol{r}')+a_{lm,l'm'}(\boldsymbol{R})\boldsymbol{N}_{l'm'}(\boldsymbol{r}')$$

where

$$a_{lm,l'm'}(\boldsymbol{R})=-\mathrm{i}^{l+l'}[l(l+1)(2l+1)(2l'+1)/l'(l'+1)]^{1/2}\times\sum_{s=|l-l'|}^{l+l'}\mathrm{i}^{-s}(2s+1)\begin{pmatrix}l&s&l'\\0&0&0\end{pmatrix}\begin{pmatrix}l&s&l'\\-m&m-m'&m'\end{pmatrix}\begin{Bmatrix}1&l&l\\s&l'&l'\end{Bmatrix}\times h_s(kR)C_{sm-m'}(\theta_R,\varphi_R)\tag{17a}$$

and $a'_{lm,l'm'}(\boldsymbol{R})$ is given by the corresponding analogue of (18). The analogue of (12) is

$$\boldsymbol{L}'_{lm}(\boldsymbol{r}) = \sum_{s,l',m'} \mathrm{i}^{l+l'-s}(-1)^m(2l'+1)(2s+1)\begin{pmatrix} l & s & l' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & s & l' \\ m & m'-m & -m' \end{pmatrix} \times h_s(kR)C_{sm-m'}(\theta_R, \varphi_R)\boldsymbol{L}_{l'm'}(\boldsymbol{r}') \qquad |\boldsymbol{r}'|<|\boldsymbol{R}|. \tag{12a}$$

Appendix

Here several properties of the vector multipole fields required in the foregoing discussion are derived. From the gradient formula we have

$$\nabla_\mu f(r)C_{lm}(\theta,\varphi) = (-1)^{l+\mu-m}(2l+1)^{-1/2}\bigg[[(l+1)(2l+3)]^{1/2}\left(\frac{1}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)C_{l+1\,m+\mu}(\theta,\varphi) \times\begin{pmatrix} l+1 & 1 & 1 \\ m+\mu & -\mu & -m \end{pmatrix} + [l(2l-1)]^{1/2}\left(\frac{l+1}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)C_{l-1\,m+\mu}(\theta,\varphi) \times\begin{pmatrix} l-1 & 1 & l \\ m+\mu & -\mu & -m \end{pmatrix}\bigg] \tag{A.1}$$

where ∇_μ are the spherical components of the gradient operator defined as in (5), and

$$\boldsymbol{\nabla} = \sum_\mu (-1)^\mu \boldsymbol{\xi}_\mu \nabla_{-\mu}.$$

Therefore

$$\boldsymbol{\nabla}\cdot f(r)\mathbf{C}^m_{l\lambda}(\theta,\varphi) = (-1)^{\lambda-1+m}(2\lambda+1)^{1/2}\sum_\mu \begin{pmatrix} \lambda & 1 & l \\ m+\mu & -\mu & -m \end{pmatrix}\nabla_{-\mu} f(r)C_{\lambda\, m-\mu}(\theta,\varphi)$$

which becomes on substituting (A.1) and exploiting the orthogonality of the 3*j*-symbols,

$$\boldsymbol{\nabla}\cdot f(r)\mathbf{C}^m_{l\lambda}(\theta,\varphi) = C_{lm}(\theta,\varphi)\left[\delta_{l,\lambda+1}\left(\frac{l}{2l+1}\right)^{1/2}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l-1}{r}\right)f(r) - \delta_{l,\lambda-1}\left(\frac{l+1}{2l+1}\right)^{1/2}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{l+2}{r}\right)f(r)\right]. \tag{A.2}$$

Similarly, starting from

$$\boldsymbol{\nabla}\times f(r)\mathbf{C}^m_{l\lambda}(\theta,\varphi) = (2\lambda+1)^{1/2}\sum_{\mu,\nu}(-1)^{\mu-1+m}\begin{pmatrix} \lambda & 1 & l \\ m-\nu & \nu & -m \end{pmatrix}\boldsymbol{\xi}_\mu\times\boldsymbol{\xi}_\nu\nabla_{-\mu}f(r)C_{\lambda\, m-\nu}(\theta,\varphi),$$

and noting that

$$\boldsymbol{\xi}_\mu\times\boldsymbol{\xi}_\nu = (-1)^{\mu+\nu}\,\mathrm{i}6^{1/2}\begin{pmatrix} 1 & 1 & 1 \\ \mu & \nu & -\mu-\nu \end{pmatrix}\boldsymbol{\xi}_{\mu+\nu}$$

we have, on substituting in (A.1) and changing dummy indices

$$i6^{1/2}\sum_{\sigma,\nu}(-1)^{\sigma+\nu+1}\xi_\sigma\begin{pmatrix}1 & 1 & 1\\ \sigma-\nu & \nu & -\sigma\end{pmatrix}\begin{pmatrix}\lambda & 1 & l\\ m-\nu & \nu & -m\end{pmatrix}$$

$$\times\Bigg[[(\lambda+1)(2\lambda+3)]^{1/2}\left(\frac{\lambda}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)C_{\lambda+1\,m-\sigma}(\theta,\varphi)$$

$$\times\begin{pmatrix}\lambda+1 & 1 & \lambda\\ m-\sigma & \sigma-\nu & -m+\nu\end{pmatrix}$$

$$+[\lambda(2\lambda-1)]^{1/2}\left(\frac{\lambda+1}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)C_{\lambda-1,m-\sigma}(\theta,\varphi)$$

$$\times\begin{pmatrix}\lambda-1 & 1 & \lambda\\ m-\sigma & \sigma-\nu & -m+\nu\end{pmatrix}\Bigg].$$

Since

$$\sum_\nu(-1)^{\nu+\sigma+1}\begin{pmatrix}\lambda' & 1 & \lambda\\ m-\sigma & \sigma-\nu & -m+\nu\end{pmatrix}\begin{pmatrix}1 & l & \lambda\\ \nu & -m & m-\nu\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ \sigma-\nu & \nu & -\sigma\end{pmatrix}$$

$$=(-1)^{\lambda'-m}\begin{Bmatrix}l & \lambda & \lambda'\\ 1 & 1 & 1\end{Bmatrix}\begin{pmatrix}l & \lambda' & 1\\ -m & m-\sigma & \sigma\end{pmatrix}$$

we get

$$\nabla\times f(r)\mathbf{C}^m_{l\lambda}(\theta,\varphi)$$

$$=i6^{1/2}(-1)^{l+\lambda+1}\Bigg[(\lambda+1)^{1/2}\left(\frac{\lambda}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\begin{Bmatrix}l & \lambda & \lambda+1\\ 1 & 1 & 1\end{Bmatrix}\mathbf{C}^m_{l\lambda+1}(\theta,\varphi)$$

$$+\lambda^{1/2}\left(\frac{\lambda+1}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\begin{Bmatrix}l & \lambda & \lambda-1\\ 1 & 1 & 1\end{Bmatrix}\mathbf{C}^m_{l\lambda-1}(\theta,\varphi)\Bigg]$$

and evaluation of the $6j$-symbol then gives

$$\nabla\times f(r)\mathbf{C}^m_{ll+1}(\theta,\varphi)=i\left(\frac{l+2}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\left(\frac{l}{2l+1}\right)^{1/2}\mathbf{C}^m_{ll}(\theta,\varphi) \tag{A.3}$$

$$\nabla\times f(r)\mathbf{C}^m_{ll-1}(\theta,\varphi)=-i\left(\frac{l-1}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\left(\frac{l+1}{2l+1}\right)^{1/2}\mathbf{C}^m_{ll}(\theta,\varphi) \tag{A.4}$$

and

$$\nabla\times f(r)\mathbf{C}^m_{ll}(\theta,\varphi)$$

$$=-i\left(\frac{l}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\left(\frac{l}{2l+1}\right)^{1/2}\mathbf{C}^m_{ll+1}(\theta,\varphi)$$

$$+i\left(\frac{l+1}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\left(\frac{l+1}{2l+1}\right)^{1/2}\mathbf{C}^m_{ll-1}(\theta,\varphi). \tag{A.5}$$

From this last result we have, on setting $f(r)=z_l(kr)$

$$k^{-1}\nabla\times\nabla\times\mathbf{r}z_l(kr)C_{lm}(\theta,\varphi)$$

$$=[l(l+1)]^{1/2}\{[(l+1)/(2l+1)]^{1/2}z_{l-1}(kr)\mathbf{C}^m_{ll-1}(\theta,\varphi)$$

$$-[l/(2l+1)]^{1/2}z_{l+1}(kr)\mathbf{C}^m_{ll+1}(\theta,\varphi)\}. \tag{A.6}$$

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